The Gravitational Vavilov–Cherenkov Effect [1]

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Abstract

In this essay we show that an uncharged black–hole moving superluminally in a transparent dielectric medium violates Hawking's area theorem. The violation is overcome through the emission of radiation. Since modes cannot emerge from the black hole itself, this radiation must originate from a collective effect in the medium, in complete analogy with the Vavilov–Cherenkov effect. However, because the black–hole is uncharged, the emission mechanism must be different. We discuss the physical origin of the effect and obtain a Newtonian estimative. Then we obtain the appropriate equations in the relativistic case and show that the field which is radiated away is a combination of gravitational and electromagnetic degrees of freedom. Possible astrophysical relevance for the detection of primordial black–holes and binary systems is discussed.

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The Classical Vavilov–Cherenkov effect

In one of the interesting twists in the development of scientific ideas, in 1904 Sommerfeld calculated the energy radiated away by a charged particle moving superluminally in the vacuum – a result that was buried with the advent of special relativity a few years later. When Cherenkov and Vavilov observed the radiation emitted by a charged particle moving superluminally in a *medium* they were unaware of Sommerfeld's calculation and the phenomenon found only the proper theoretical explanation by the hands of Tamm and Frank in 1937 [1, 2].

The phenomenon can be easily understood from heuristic considerations. Think of a system moving inertially with velocity $\vec{v}$ through a medium of refraction index n. As N particles are emitted by the system , its internal energy changes by the amount:

$$\Delta M = -N\hbar(\omega - \vec{v} \cdot \vec{k}) \tag{1}$$

where ω and $\vec{k}$ are the frequency and wave vector of the emitted photons, and back-reaction effects were neglected. Because $\vec{v} \cdot \vec{k} = \omega n v \cos\theta$, where θ is the angle formed between the directions of propagation of the photon and that of motion of the particle,

$$\Delta M = -N\hbar\omega(1 - nv\cos\theta) \tag{2}$$

Similarly, the absorption of N' photons is followed by a change in the system's internal energy:

$$\Delta M' = N'\hbar\omega(1 - nv\cos\theta) \tag{3}$$

An elementary particle is a structureless system, thus $\Delta M = \Delta M' = 0$. If the particle moves subluminally then $nv < 1$ and it can neither emit nor absorb a photon from the environment. In contrast, if it moves fast enough such that the *Ginzburg–Frank* condition $nv > 1$ is met, then it can either emit or absorb photons provided they lie on the conical surface

$$\cos\theta_0 = \frac{1}{nv} \tag{4}$$

What is actually observed is light emitted from the whole interior of the Cherenkov cone. The reason for this is that the charged particle in a dielectric cannot be thought of as structureless as it creates a polarization cloud that is dragged alongside its motion. In a process involving absorption and emission of quanta in a given field mode, the total entropy change of polarization cloud is

$$\delta S_{\text{cloud}} = T^{-1}(N' - N)(1 - nv\cos\theta) \tag{5}$$

where T is the temperature of the medium, here assumed homogeneous. The entropic bookkeeping should also include the entropy of the absorbed and emitted photons. For intense radiation the entropy change in the radiation behaves as $\delta S_{\rm rad} \approx \log(N'/N)$, which is much smaller than the entropy change in the polarization cloud and can be neglected [3]. Thus the second law requires that

$$(N' - N)(1 - nv\cos\theta) > 0\,. \tag{6}$$

Accordingly, in the absence of incoming radiation $N' = 0$, the second law says that modes lying inside the Cherenkov cone are spontaneously emitted. *En passant*, we mention a new effect discussed recently in which there is some incoming radiation. In such a circumstance the second law imposes $N > N'$, that is to say, the outgoing radiation is amplified for modes lying inside the Cherenkov cone, an effect so far not observed [3]. The present discussion makes transparent the fact that the light emission mechanism is a collective effect of the medium.

On a more formal basis, the Cherenkov radiation is derived from Maxwell's equations for a moving particle as a source [4]:

$$\begin{aligned} \vec{\nabla}\cdot\vec{B} &= 0 & \vec{\nabla}\times\vec{B} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} &= 0 \\ \vec{\nabla}\cdot\vec{D} &= 4\pi\, Q\,\delta^3(\vec{x}-\vec{v}t) & \vec{\nabla}\times H - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} &= 4\pi\vec{v}\, Q\,\delta^3(\vec{x}-\vec{v}t), \end{aligned} \tag{7}$$

with $\vec{D} = \hat{\varepsilon}\vec{E}$, where $\hat{\varepsilon}$ is the dielectric operator (this expression stands for the Fourier transform of $\vec{D}(\omega) = \varepsilon(\omega)\vec{E}(\omega)$) and $\vec{B} = \vec{H}$. The stopping power dF, the energy radiated per unit of path length per frequency interval that follows from these equations is [4]

$$dF = -\frac{ie^2}{\pi}d\omega\left(\frac{1}{c^2} - \frac{1}{\varepsilon v^2}\right)\sum\omega\int\frac{d\xi}{\xi} \tag{8}$$

where the summation refers to the two branches $\omega = \pm\,|\omega|$ and $\xi \equiv q^2 - \omega^2(\varepsilon/c^2 - 1/v^2)$ with q standing for the transverse momentum. Although the medium is transparent it has always a (very) small imaginary part $\Im(\omega) \gtrless 0$ for $\omega \gtrless 0$, a consequence of causality. Because for $v > c/\varepsilon$ the poles are located beneath/above the real axis for $\omega \gtrless 0$ the contour integral is indented such that it is closed in the upper/lower plane for $\omega \gtrless 0$. In this case, upon summing both contributions the real parts drop out and the final result is [4]

$$dF = \frac{e^2}{c^2}\left(1 - \frac{c^2}{n^2v^2}\right)\omega d\omega \tag{9}$$

Note that for $v < c/\varepsilon$, the integral vanishes rendering crystal clear the relevance of the Frank–Ginzburg condition.

Uncharged black–holes moving superluminally

The discussion in the previous section on the amplification of incoming radiation for modes lying within the Cherenkov cone is reminiscent of superradiant amplification of radiation by charged/rotating black holes and brings to our mind the *gedanken* experiment in which the charged particle is replaced by a neutral black hole moving inertially but superluminally through a dielectric but transparent medium with refraction index $n(\omega) > 1$. In order to avoid accretion of matter by the hole, imagine a solid dielectric in which a narrow straight channel is drilled through which the hole traverses. What happens? Since the black hole is uncharged the sole effect is a tidal wake inside the medium that propagates alongside the black hole. Is that all? That something novel happens here is a direct consequence of Hawking's area theorem [5]. The black hole area change due to the absorption of N photons from the medium in a mode $(\omega, \vec{k})$ is

$$\Delta A_{\rm bh} = 32\pi M \Delta M = 32\pi\omega\, N\,(1 - nv\cos\theta)\hbar \tag{10}$$

Accordingly, the absorption of photons in modes lying inside the Cherenkov cone leads to a violation of the area theorem! In vacuum the unique scenario which does not violate Hawking's area theorem is the spontaneous emission ($N < 0$) of quanta in Cherenkov modes. We reach to the conclusion that *uncharged black–holes moving superluminally in a medium spontaneously emit radiation.* This violation was recently discussed in [3] where the emission mechanism was made transparent. Since waves cannot classically emerge from within the hole we must look for their source in a collective effect in the medium, like in Vavilov–Cherenkov effect, but must differ from that one because the black–hole is uncharged. Clearly in the conversion of kinetic energy to waves, gravitation must play the pivotal role. *Because gravity pulls on the positively charged nuclei in the dielectric stronger than on the enveloping electrons an electric field develops inside the atom neutralizing the gravitational pull and preventing ionization. It is in this way that an electrical polarization of the dielectric emerges. The polarization cloud that dresses the black–hole and moves alongside the black-hole should be viewed as the true source of radiation.* This argument makes transparent the fact that the black hole character of the moving source is immaterial here as what really counts is its gravitational pull, so long as both are much smaller than the channel's width. An ordinary object with the same mass would have similar effect as a black hole.

We can render this intuitive picture quantitative by noting that in the Newtonian

limit the induced polarization $\vec{E}_0$ is the field that balances the gravitational pull $\vec{g}$:

$$e\vec{E} = -\delta\mu\,\vec{g}\,, \tag{11}$$

where $\delta\mu \approx Am_p$ is the nucleus–electron mass difference (A is the mass number of the atoms, m_p the proton's mass), and $e > 0$ the unit of charge. From the gravitational Poisson equation it follows that

$$\vec{\nabla}\cdot\vec{E}_0 = 4\pi GM(\delta\mu/e)\delta(\vec{r}-\vec{r}_0) \tag{12}$$

where $\vec{r}_0$ denotes the momentary black hole position. The electric field accompanying the black hole is thus that of a pointlike charge $Q \equiv GAMm_p/e$. Note that we need $|\vec{v}|$ to be sufficiently large for the Ginzburg–Frank condition to hold and consequently we must go beyond the Newtonian approximation.

The Gravitational Vavilov–Cherenkov Effect

In order to evade from complications stemming from matter accretion, we drilled a channel in the dielectric whose width is very much larger than the Schwarzschild radius. Under this circumstance we can consider Einstein's theory in its linearized form. We focus now our attention on a single atom in the dielectric. Let $\vec{u}$ represent the mean velocity of the electrons and the nucleus of such an atom averaged over a time scale large enough to neglect internal motion within the atom but not too large to disregard thermal or oscillatory motions within the dielectric (phonon waves). Clearly $|\vec{u}| << 1$ (hereafter we set $c = 1$). Thus, up to the first order in the velocity the gravitational pull differential in the atom is:

$$F^i_{\text{grav}} = \delta\mu\frac{d^2x^i}{dt^2} = -\delta\mu\left(\Gamma^i_{00} + 2\Gamma^i_{0j}u^j\right) \tag{13}$$

As we discussed already, electromagnetic fields develop inside the atom in order to prevent ionization. Because the electromagnetic force

$$\vec{F}_{\text{elect}} = e\left(\vec{E}_0 + \vec{u}\times\vec{B}_0\right) \tag{14}$$

must balance the gravitational pull, we identify

$$\begin{aligned} eE^i_0[\vec{u}] &= \delta\mu\left(h_{0i,j} - \frac{1}{2}h_{00,i} + h_{ij,0}u^j\right)\,; && (15)\\ e\epsilon_{ijk}B^k_0[\vec{u}]u^j &= \delta\mu\left(h_{i0,j} - h_{j0,i}\right)u^j\,, && (16)\end{aligned}$$

where the linearized version of the Christoffel symbols was used. This decomposition is unique and dictated by the fact that the first line is associated to a force that does some work while the second line it does not. For non-relativistic motion inside the dielectric, we can omit the velocity dependent term in E^i_0, in which case:

$$
\begin{aligned}
E^i_0 &= \frac{\delta\mu}{e}(h_{0i,0} - \frac{1}{2}h_{00,i})\,; \\
B^i_0 &= \frac{2\delta\mu}{e}\epsilon_{ijk}h_{0j,k}\,.
\end{aligned} \tag{17}
$$

In the gauge $\overline{h}^{\mu\nu}_{,\nu} = 0$, the linearized Einstein's equations read

$$
\Box\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}\,, \tag{18}
$$

where $T_{\mu\nu}$ is the energy momentum of a source of proper mass M and velocity v^μ

$$
T_{\mu\nu} = M v_\mu v_\nu \gamma^{-1}\delta^3(\vec{x} - \vec{v}t)\,. \tag{19}
$$

For constant velocity of the source, linearized gravity is equivalent to a theory involving a single scalar gravitational potential ψ :

$$
h_{\mu\nu} = 4(v_\mu v_\nu + \frac{1}{2}\eta_{\mu\nu})\psi\,, \tag{20}
$$

that satisfies

$$
\Box\psi = -4\pi M\gamma^{-1}\delta^3(\vec{x} - \vec{v}t). \tag{21}
$$

In terms of this field, the gauge condition translates into

$$
\frac{\partial\psi}{\partial t} + \vec{v}\cdot\vec{\nabla}\psi = 0\,, \tag{22}
$$

where $(\vec{v})^i = v^i/v^0$ and $\gamma = v_0^2$ is the Lorentz factor. In the three dimensional vector notation the polarization reads:

$$
\begin{aligned}
\vec{E}_0 &= \frac{4\delta\mu}{e}\left[\gamma^2\vec{v}\dot{\psi} - \frac{1}{2}(\gamma^2 - \frac{1}{2})\vec{\nabla}\psi\right] \\
\vec{B}_0 &= \frac{8\delta\mu\gamma^2}{e}\vec{v}\times\vec{\nabla}\psi
\end{aligned} \tag{23}
$$

Like in pyroelectric media, the electromagnetic fields are the sum of the polarization given by eq. (23) and the perturbations $(\vec{e}, \vec{b})$, $\vec{E} = \vec{E}_0 + \vec{e}$; $\vec{B} = \vec{B}_0 + \vec{b}$. In terms of the new fields and in the absence of external charges or currents, the first pair of Maxwell's equations read[4]

[4]In what follows, $\frac{1}{\tilde{\varepsilon}}f(t) = \frac{1}{\tilde{\varepsilon}}\int \tilde{f}(\omega)e^{-i\omega t}d\omega = \int \frac{\tilde{f}(\omega)}{\varepsilon(\omega)}e^{-i\omega t}d\omega$

$$\vec{\nabla}\cdot\vec{d} \;=\; \frac{4\hat{\varepsilon}\delta\mu}{e}\left[\gamma^2\ddot{\psi}+\frac{1}{2}(\gamma^2-\frac{1}{2})\nabla^2\psi\right]\,; \tag{24}$$

$$\vec{\nabla}\times\vec{b}-\frac{\partial\vec{d}}{\partial t} \;=\; \frac{4\hat{\varepsilon}\delta\mu}{e}\left[\gamma^2\vec{v}\ddot{\psi}-\frac{1}{2}(\gamma^2-\frac{1}{2}\vec{\nabla}\dot{\psi})-\frac{2\gamma^2}{\hat{\varepsilon}}(\vec{v}\nabla^2\psi+\vec{\nabla}\dot{\psi})\right], \tag{25}$$

where $\vec{d}=\hat{\varepsilon}\vec{e}$. Inspired by the Newtonian discussion, we wished to interpret the right hand side of these equations as the effective charge density and current of the polarization cloud. The equation $\vec{\nabla}\cdot\vec{B}=0\rightarrow\vec{\nabla}\cdot\vec{b}=0$, lends support to this strategy. Unfortunately, the remaining equation

$$\vec{\nabla}\times\vec{e}+\frac{\partial\vec{b}}{\partial t}=-\frac{4\gamma^2\delta\mu}{e}\,\vec{v}\,\times\,\vec{\nabla}\dot{\psi} \tag{26}$$

is not homogeneous preventing such and identification. We overcome this hurdle by defining a new pair of fields $(\vec{\epsilon},\vec{\beta})$ which combine together electromagnetic and gravitational degrees of freedom:

$$\vec{\epsilon} \;=\; \vec{e}-\frac{4\gamma^2\delta\mu}{e}\,\vec{\sigma} \tag{27}$$

$$\vec{\beta} \;=\; \vec{b}-\frac{4\gamma^2\delta\mu}{e}\,\vec{\theta} \tag{28}$$

where

$$\vec{\sigma} \;=\; \left(\frac{1}{2}-\frac{1}{4\gamma^2}+\frac{2}{\hat{\varepsilon}}\right)\vec{\nabla}\psi+\frac{2-\hat{\varepsilon}}{2}\dot{\psi}\,\vec{v} \tag{29}$$

$$\vec{\theta} \;=\; 2\frac{1-\hat{\varepsilon}}{\hat{\varepsilon}}\,\vec{v}\,\times\,\vec{\nabla}\psi \tag{30}$$

In terms of these new fields the offending term in eq.(26) is removed and the field equations attain Maxwell's form

$$\vec{\nabla}\cdot\vec{\delta} \;=\; 4\pi\,\overline{Q}\,\delta^3(\vec{x}-\vec{v}t)\,; \tag{31}$$

$$\vec{\nabla}\times\vec{\beta}-\frac{\partial\vec{\delta}}{\partial t} \;=\; 4\pi\,\overline{Q}\,\vec{v}\delta^3(\vec{x}-\vec{v}t)+\vec{j}\,; \tag{32}$$

$$\vec{\nabla}\cdot\vec{\beta} \;=\; 0\,; \tag{33}$$

$$\vec{\nabla}\times\vec{\epsilon}+\frac{\partial\vec{\beta}}{\partial t} \;=\; 0\,, \tag{34}$$

where $\vec{\delta}=\hat{\varepsilon}\vec{\epsilon}$,

$$\overline{Q}=\frac{8M\delta\mu}{e^2M_p^2}\,e\,, \tag{35}$$

with e the elementary charge and

$$\vec{j} = -\frac{(1-\hat{\varepsilon})\,\gamma\,\overline{Q}}{4\pi M\epsilon}\left(\vec{v}\,\nabla^2\psi + \vec{\nabla}\dot{\psi}\right)\,, \tag{36}$$

a divergence free current (by virtue of equation (22)).

Our new set of equations differ from eqs.(7) in that they have an additional current $\vec{j}$. Nevertheless, this extra term is not responsible for additional work on the moving particle because of $\vec{j}\cdot\vec{v}=0$ [see eq.(22)] and the linearity of the differential equations. Put in other words, this term is not responsible for energy dissipation and the formula for the stopping power [eq.(8)] applies to these equations as well under the replacement[5] $Q \to \overline{Q}$.

Summarizing our results we say that whenever a gravitating object moves superluminally in a dielectric medium and the Frank–Ginzburg condition is met, then the composite system (object+medium) will radiate an admixture of gravitational and electromagnetic degrees of freedom. This is what we call the gravitational Vavilov–Cherenkov effect. It is important to remark here that the non-relativistic approach missed a Lorentz factor in the effective charge $\overline{Q}$ of the moving source. In turn, this factor is responsible to an enhancement of the energy radiated away by the very same factor squared [see eq.(9)]. Could the effect have practical importance or is it merely of academic interest? Because Q/e is about $10^3 A$ times the gravitational radius of the hole measured in units of the classical radius of the *electron*, a fast 10^{15} g primordial black hole moving in a suitable dielectric would radiate just like an equal fast particle bearing $\sim 10^3 A$ elementary charges. This is relevant for the experimental search for primordial black holes. Furthermore the Newtonian approach indicates an analog of synchrotron radiation for a gravitating system in circular motion. This might be of astrophysical importance in context of binary systems.

As a concluding remark, we emphasize that our argument involves a big assumption, that the dielectric has time to relax to form the above compensating field. Such relaxation does occur for sufficiently small $|\mathbf{v}|$, but since we need $|\mathbf{v}|$ to be sufficiently large for the Ginzburg–Frank condition to hold, stringent conditions are required of

[5]One could have wondered whether this application of the stopping power formula is legitimate because a proper evaluation of the power dissipated by the source should involve both electromagnetic and gravitational 'frictions', something we did not give its due care. Some reasoning reveals that after all our result is correct. Because the above 'electromagnetic equations' can be derived from a variational principle involving a vector potential α_μ with an interaction term $\overline{Q}\int \alpha_\mu dx^\mu$, the variation of this term with respect to the path of a test particle path yields a Lorentz force $\vec{F}_{\rm friction} = \overline{Q}(\vec{\epsilon}+\vec{v}\times\vec{\beta})$. Accordingly, gravitation and electromagnetic 'frictions' are automatically taken care of, validating our procedure.

the dielectric (high n and fast relaxation). Thus, the form of the dissipation could differ considerably from the one here discussed for media that do not relax fast enough.

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